2014 BC #6
(no calculator)

$$f(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n \text{ for } |x-1|<R \text{ where } R \text{ is the radius of convergence}$$

$$f(x)=\frac{2}{1}(x-1)-\frac{2^2}{2}(x-1)^2+\frac{2^3}{3}(x-1)^3-\cdots$$

(a) Using the Ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^{n+1}}{n+1}\bullet\frac{n}{2^n(x-1)^n}\right|=\lim_{n\to\infty}\left|\frac{2^n2^1(x-1)^n(x-1)^1}{n+1}\bullet\frac{n}{2^n(x-1)^n}\right|=$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)|2(x-1)|=|2(x-1)|<1 \Rightarrow |x-1|<\frac{1}{2} \Rightarrow \boxed{R=\frac{1}{2}}$$

The root test would have worked here also.

(b) Taking the derivative of $f(x)$ above:

$$f'(x)=\boxed{2-2^2(x-1)^1+2^3(x-1)^2-\cdots}$$

$$\text{general term}=\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}2^n(x-1)^{n-1}}$$

(c)

Since f' is geometric, then $f'=\frac{a_1}{1-r}$ where the first term, $a_1=2$ and the common ratio, $r=-2(x-1)$. So $\boxed{f'(x)=\frac{2}{1-(-2(x-1))}}=\frac{2}{2x-1}$

Now, $f(x)=\int\frac{2}{2x-1}dx=\ln|2x-1|+C \Rightarrow$

Since $f(1)=0$ (in the original series above), $0=0+C\Rightarrow C=0$

and absolute value can be removed - since $R=\frac{1}{2}$, then $\frac{1}{2}<x<\frac{3}{2}$

and $2x-1>0$ on this interval.

So, $\boxed{f(x)=\ln(2x-1)}$